STEADY HEAT TRANSFER IN LAMINAR FLUID FLOW AT VARIOUS PECLET NUMBERS

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An approximate analytical method is proposed for solving the convective heat-transfer equations; the method is based on series expansion in terms of the eigenfunctions of the heat-conduction equation.

Methods of solving the problem of steady heat transfer in a laminar flow are basically numerical, and may be conventionally divided into two basic types: finding the eigenvalues and eigenfunctions by series expansion with respect to the transverse coordinate [1, 2] and using various versions of the finite-difference method [1-5].

The analytical method proposed in the present work for calculating the temperature field and heat flux with arbitrary values of the Peclet number Pe may be used for both pellicular flows and flows in channels.

It is assumed below that the boundary conditions and velocity of the fluid $v = v_x(y)$ do not change along the flow.

With constant fluid properties, the given problem may be described by the following equation

$$v\frac{\partial T}{\partial x} = a\Delta T$$

and the boundary conditions

$$x > 0:\ \left(\alpha_i T + \beta_i \frac{\partial T}{\partial y}\right)\Big|_{y=y_i} = \gamma_i\ (i = 1,\ 2),\qquad x = 0:\ T = T_0(y), \tag{1}$$

which give conditions of the first, second, or third kind with definite values of the coefficients α_i, β_i, γ_i.

Homogeneous boundary conditions in terms of y are more convenient than Eq. (1) for solution. They may be derived by introducing the auxiliary function $T_1(x, y)$ satisfying the equation for $T(x, y)$ and the boundary conditions in Eq. (1). The function $T_1(x, y)$ may be regarded as the fluid temperature at a large distance from the input to the heat-transfer region. It is simple to obtain an expression for $T_1(x, y)$ as a rule, and therefore it is assumed to be known. Subtracting T_1 from T, a new function satisfying boundary conditions homogeneous in y is obtained.

Converting to dimensionless quantities — $\Theta = (T - T_1)/T^*$, $\hat{v} = vS/Q$, $\hat{x} = x/h$, $\hat{y} = y/h$, $\hat{\beta}_i = \beta_i/h$, $Pe = Qh/Sa$, $d\hat{s} = ds/S$ — and omitting the tildes for the sake of convenience, the result obtained is

$$\mathrm{Pe}\, v\frac{\partial \Theta}{\partial x} = \Delta\Theta, \tag{2}$$

$$\left(\alpha_i\Theta + \beta_i\frac{\partial \Theta}{\partial y}\right)\Big|_{y=y_i} = 0,\quad \Theta|_{x=0} = \Theta_0(y).$$

The essence of this method is to expand the desired solution of Eq. (2) in series in eigenfunctions of the heat-conduction problem, with the same boundary conditions, and then to determine the coefficients of these functions and also the spectrum of the problem.

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As $Pe \to 0$, the left-hand side of Eq. (2) may be neglected, and the equation reduces to the heat-conduction equation, the solution of which $\Theta^{(0)}(x, y)$ is regarded as known [6]. In general form, $\Theta^{(0)}(x, y)$ is determined as the sum of the series

$$\Theta^{(0)}(x, y) = \sum_{n=1}^{\infty} C_n \varphi_n(y) \exp(-\nu_n x), \tag{3}$$

where ν_n and $\varphi_n(y)$ are the eigenvalues and eigenfunctions of the boundary problem

$$L\varphi_n + \nu_n^2 \varphi_n = 0, \; L = \Delta - \frac{\partial^2}{\partial x^2}, \; \left(\alpha_i \varphi_n + \beta_i \frac{d\varphi_n}{dy}\right)\Big|_{y=y_i} = 0.$$

The set of eigenfunctions φ_n has the properties of completeness and orthogonality [7] and, with the corresponding normalization, it follows that $\int_\Omega \varphi_n \varphi_m ds = \delta_{nm}$, where $\int_\Omega ds$ is the integral over the cross section of the flow Ω.

From the condition $\Theta^{(0)}|_{x=0} = \Theta_0(y)$ and the finiteness of $\Theta^{(0)}(x, y)$ as $x \to \infty$, it follows that $C_n = \int_\Omega \Theta_0 \varphi_0 ds, \; \nu_n > 0$.

The solution of Eq. (2) with arbitrary Pe may also be expanded in series in terms of φ_n

$$\Theta(x, y) = \sum_{n=1}^{\infty} f_n(x) \varphi_n(y), \; f_n(x) = \int_\Omega \Theta(x, y) \varphi_n(y) ds. \tag{4}$$

Substituting Eq. (4) into Eq. (2), after simple transformations, a system of equations for determining $f_n(x)$ is obtained

$$\frac{d^2 f_n}{dx^2} - \nu_n^2 f_n = \mathrm{Pe} \sum_{m=1}^{\infty} a_{nm} \frac{df_m}{dx}, \; a_{nm} = \int_\Omega v \varphi_n \varphi_m ds.$$

The boundary conditions $f_n(0) = C_n$, $f_n(\infty) < \infty$ are added to this system.

The system obtained is a system of linear homogeneous differential equations with constant coefficients. Assuming that there are no multiple roots of the characteristic equation, its solution may be written in series form

$$f_n(x) = \sum_{l=1}^{\infty} g_{nl} \exp(-k_l x).$$

After substitution into the corresponding equation and boundary conditions, a system of algebraic equations for determining the coefficients g_{nl} and the spectrum k_l is obtained

$$g_{nl}(\nu_n^2 - k_l^2) = \mathrm{Pe}\, k_l \sum_{m=1}^{\infty} a_{nm} g_{ml}, \tag{5}$$

$$\sum_{l=1}^{\infty} g_{nl} = C_n, \; k_n > 0. \tag{6}$$

Thus, the problem in Eq. (2) reduces to a system of algebraic equations. Since the number of equations and unknowns in Eqs. (5) and (6) is infinite and the system is nonlinear, it is impossible to obtain its accurate solution in general form.

In the particular case when $v = \mathrm{const} = 1$ (core flow), the accurate solution of Eqs. (5) and (6) takes the form

$$g_{nl} = C_n \delta_{nl}, \; k_n = \frac{1}{2}\left(\sqrt{\mathrm{Pe}^2 + 4\nu_n^2} - \mathrm{Pe}\right).$$

The corresponding temperature profile is determined by the expression

$$\Theta = \sum_{n=1}^{\infty} C_n \varphi_n(y) \exp(-k_n x), \tag{7}$$

i.e., differs from the solution of the heat-conduction problem in Eq. (3) only by the spectrum $\{k_n\}$.

To obtain an approximate solution with arbitrary v(y), several properties of the matrix a_{nm} are considered. The flow is assumed below to be unidirectional: v(y) ≥ 0 when $y \in \Omega$).

Then it follows from $\int_\Omega \varphi_n^2 ds = \int_\Omega v ds = 1$ that $a_{nm} = \int_\Omega v\varphi_n^2 ds \sim 1$.

Since, when n > 1, $\varphi_n(y)$ is a sign-variable function and the number of its zeros in Ω is proportional to n, the function $\Phi_{nm} = v\varphi_n\varphi_m$ is also sign-variable when n ≠ m. The number of regions in which $\Phi_{nm} > 0$ is approximately the same as the number of regions where $\Phi_{nm} < 0$. It is natural to assume that $|a_{nm}| \ll 1$, i.e., the nondiagonal matrix elements a_{nm} are much less than the diagonal elements, while $|a_{nm}|$ decreases with increase in $|n - m|$.

It follows from the properties of a_{nm} and from Eqs. (5) and (6) that, in each column and row of the matrix g_{nm}, the diagonal element is the greatest (in modulus).

Taking account of the elements adjacent to the diagonal elements a_{nm} and g_{nm}

$$\begin{gathered} g_{nm} = g_{nn}\delta_{nm} + g_{nn-1}\delta_{n-1m} + g_{nn+1}\delta_{n+1m} + g_{nn-1}\delta_{nm-1} + g_{n+1n}\delta_{nm+1}, \\ a_{nm} = a_{nn}\delta_{nm} + a_{nn-1}(\delta_{n-1m} + \delta_{nm-1}) + a_{nn+1}(\delta_{n+1m} + \delta_{nm+1}), \end{gathered} \tag{8}$$

the result obtained after substituting Eq. (8) into Eqs. (5) and (6) is

$$g_{n+1n} = \mathrm{Pe}\, k_n g_{nn} F_n a_{nn}, \tag{9}$$

$$g_{n-1n} = -\mathrm{Pe}\, k_n g_{nn} F_{n-1} a_{nn}, \tag{10}$$

$$g_{nn} = C_n + \mathrm{Pe}\,(C_{n+1}k_{n+1}a_{n+1n+1}F_n - C_{n-1}k_{n-1}a_{n-1n-1}F_{n-1}), \tag{11}$$

$$k_n = \frac{1}{2(1+\mathrm{Pe}^2 D_n)}\left(\sqrt{\mathrm{Pe}^2(a_{nn}^2 + 4\nu_n^2 D_n) + 4\nu_n^2} - \mathrm{Pe}\, a_{nn}\right), \tag{12}$$

$$F_n = \frac{a_{n+1n}}{a_{nn}\nu_{n+1}^2 - a_{n+1n+1}\nu_n^2}, \quad D_n = a_{nn}(a_{nn+1}F_n - a_{nn-1}F_{n-1}).$$

Hence an expression for the temperature profile is obtained

$$\Theta(x, y) = \sum_{n=1}^{\infty} g_{nn}(\varphi_n + \mathrm{Pe}\, k_n a_{nn}[F_n\varphi_{n+1} - F_{n-1}\varphi_{n-1}]) \exp(-k_n x). \tag{13}$$

As Pe → ∞, as is known, a new arbitrary coordinate $x_1 = x/\mathrm{Pe}$ may be introduced. In this case, Eq. (2) transforms to the heat-transfer equation in the absence of heat conduction in the direction of the flow

$$v\frac{\partial\Theta}{\partial x_1} = L\Theta,$$

the solution of which is obtained from Eqs. (9)-(13)

$$\Theta(x_1, y) = \sum_{l=1}^{\infty} \psi_l(y) \exp(-\lambda_l x_1).$$

Here λ_l and $\psi_l(y)$ are the eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$-\lambda_l v\psi_l = L\psi_l, \quad \left.\left(\alpha_i\psi_l + \beta_i\frac{d\psi_l}{dy}\right)\right|_{y=y_i} = 0.$$

In the given approximation, the expressions for λ_l and ψ_l take the form

$$\lambda_l = \lim_{\mathrm{Pe}\to\infty} \mathrm{Pe}\, k_l = \frac{1}{2D_l}\left(\sqrt{a_{ll}^2 + 4\nu_l^2 D_l} - a_{ll}\right),$$

$$\psi_l = g_{ll}(\varphi_l + a_{ll}[F_l\varphi_{l+1} - F_{l-1}\varphi_{l-1}]\lambda_l).$$

Returning to Eq. (2) with arbitrary Pe, it may be shown that, taking account of the matrix elements farther from the diagonal elements a_{nm} and g_{nm}, the solution of Eq. (2) is obtained with any specified accuracy

$$\Theta = \sum_{l=1}^{\infty} \exp(-k_l x) \sum_{n=l-M}^{n=l+M} g_{nl}\varphi_n(y).$$

The relative error in this case is no more than $|a_{1M+1}|\mathrm{Pe}k_1/\nu_1^2$. Since it is practically impossible to calculate the temperature $\Theta(x, y)$ in the form of an infinite series, there arises the question of how many terms must be taken into account in order to achieve the required accuracy. It may be shown that, if the sequence of coefficients $|C_l|$, which is determined by the temperature $\Theta_0(y)$, is monotonically convergent (which corresponds to a sufficiently broad class of problems), the relative error associated with taking account only of the first N terms of the series is of order $|C_{N+1}/C_1|\exp((-k_{N+1} - k_1)x)$.

The results of calculating the maximum relative error for flow through a plane channel with a constant wall temperature and T_0 = const are shown in Table 1.

It must also be taken into account that, for sufficiently large l ($\nu_l \gg \mathrm{Pe}$), convective heat transfer may be neglected. Then

$$g_{ll}/C_l = k_l/\nu_l = 1 + O\left(\frac{\mathrm{Pe}}{\nu_l}\right),$$

and hence the eigenvalues and eigenfunctions of Eq. (2) may be regarded as coinciding with the corresponding eigenvalues and eigenfunctions of the heat-conduction problem, which allows the volume of computations to be significantly reduced.

Knowing the temperature profile, the Nusselt number Nu may be found; it is determined from the mean-flow-rate temperature $\bar{\Theta}$

$$\mathrm{Nu}_i(x) = -\frac{1}{\bar{\Theta}}\left(\frac{\partial\Theta}{\partial y}\right)\bigg|_{y=y_i} = -\frac{\sum\limits_{n,l} g_{n,l}\left(\frac{d\varphi_n}{dy}\right)_{y=y_i}\exp(-k_l x)}{\sum\limits_{n,l} g_{nl}\bar{\varphi}_n\exp(-k_l x)}.$$

$$\bar{\Theta} = \int_{\Omega} v\Theta ds, \quad \bar{\varphi}_n = \int_{\Omega} v\varphi_n ds,$$

The limiting Nusselt number $\mathrm{Nu}_{i\infty} = \lim\limits_{x\to\infty}\mathrm{Nu}_i(x)$ is determined by the coefficients of $\exp(-k_1 x)$

$$\mathrm{Nu}_{i\infty} = -\frac{\sum\limits_{n} g_{n1}\left(\frac{d\varphi_n}{dy}\right)\Big|_{y=y_i}}{\sum\limits_{n} g_{n1}\bar{\varphi}_n}.$$

To compare this new method with the well-known version, the results of calculating $\mathrm{Nu}_\infty(\mathrm{Pe})$ for a case that has been well investigated are given: for flow through a plane channel with a constant wall temperature.

Curves of Nu_∞ as a function of Pe are shown in Fig. 1: 1) taking account only of the diagonal matrix elements a_{nm} and g_{nm}; 2) taking account also of those closest to the diagonal; 3) numerical calculation [1], for comparison. As is evident from Fig. 1, the difference between curves 2 and 3 is slight, and hence taking account of the matrix elements adjacent to the diagonal elements a_{nm}, g_{nm} allows a solution sufficiently close to the accurate result to be obtained (relative error less than 2%).

In conclusion, as an example of the use of the given method, consider steady heat transfer between a liquid layer running down an inclined plane in laminar conditions and a gas flow around it. The tangential stress at the free surface is determined by the dimensionless parameter τ.

The liquid velocity is

$$v(y) = \frac{3Q_1}{h(1+\tau)}\left[\left(\frac{2}{3}\tau+1\right)\frac{y}{h}-\frac{y^2}{2h^2}\right].$$

The boundary conditions for the temperature are as follows

$$T|_{x=0} = T_0,\ T|_{y=h} = T_1,\ \left.\frac{\partial T}{\partial y}\right|_{y=0} = 0.$$

The eigenfunctions φ_n and eigenvalues ν_n of the heat-conduction problem in this case take the form

$$\varphi_n = \sqrt{2}\cos\nu_n\frac{y}{h},\quad \nu_n = \frac{2n-1}{2}\pi.$$

The coefficients required to calculate the dimensionless temperature $\Theta = (T - T_1)/(T_0 - T_1)$ and the Nusselt number are determined by the expressions

$$C_n = \frac{2\sqrt{2}(-1)^{n+1}}{(2n-1)\pi};\ \bar{\varphi}_n = \frac{C_n}{1+\tau}\left[(3+2\tau)\left(1-\frac{C_n}{\sqrt{2}}\right)-\frac{3}{2}(1-C_n^2)\right];$$

$$\left.\frac{d\varphi_n}{dy}\right|_{y=h} = \frac{\sqrt{2}\nu_n}{h}(-1)^n;\ a_{nn} = 1-\frac{1}{4\nu_n^2}\frac{3+4\tau}{1+\tau};$$

$$n \neq m: n+m \quad \text{even},\ a_{nm} = -\frac{1}{\pi^2(1+\tau)}\left[\frac{3}{(n-m)^2}-\frac{3+4\tau}{(n+m-1)^2}\right];$$

$$n+m \quad \text{odd},\quad a_{nm} = -\frac{1}{\pi^2(1+\tau)}\left[\frac{3+4\tau}{(n-m)^2}-\frac{3}{(n+m-1)^2}\right].$$

In Eq. (8), determining g_{nm} and k_n from Eqs. (9)-(12), the desired functions $\Theta(x, y)$ and Nu(x) are obtained

$$\Theta = \sum_{n=1}^{\infty}(g_{nn}\varphi_n + g_{n-1n}\varphi_{n-1} + g_{n+1n}\varphi_{n+1})\exp\left(-k_n\frac{x}{h}\right),$$

$$\mathrm{Nu} = \frac{\pi^2}{4}\times$$

$$\times\frac{\sum_{n=1}^{\infty}(-1)^{n+1}[(2n-1)g_{nn}-(2n-3)g_{n-1n}-(2n+1)g_{n+1n}]\exp\left(-k_n\frac{x}{h}\right)}{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{C_n(2n-1)}(g_{nn}\bar{\varphi}_n+g_{n-1n}\bar{\varphi}_{n-1}+g_{n+1n}\bar{\varphi}_{n+1})\exp\left(-k_n\frac{x}{h}\right)}.$$

The explicit form of $g_{nm}(\mathrm{Pe}, \tau)$ and $k_n(\mathrm{Pe}, \tau)$ is not given here for reasons of space.

TABLE 1. Error for Various M and N when x = 0 and Pe → ∞, %

M	N			
	5	10	25	50
0	21	16	13	12
1	12	7	4	3
2	10	5	2	1

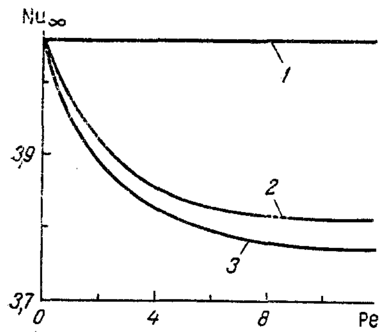

Fig. 1. Dependence of Nu_∞ on Pe for flow over a plane channel with a constant wall temperature.

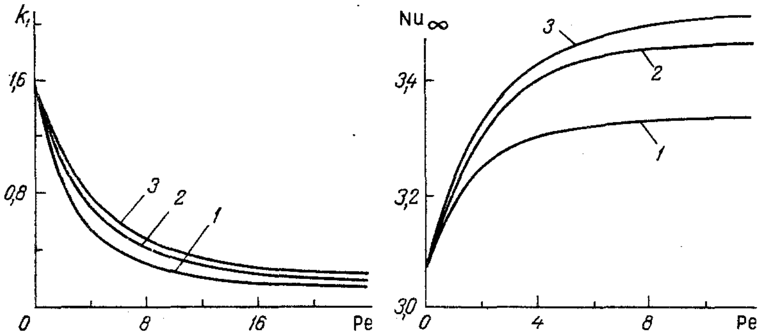

Fig. 2. Dependence of k_1 and Nu_∞ on Pe for flow over an inclined plane: 1) $\tau = -0.75$; 2) $\tau = 0$; 3) $\tau \to \infty$.

Curves of $k_1(Pe, \tau)$ and $Nu_\infty(Pe, \tau)$ are shown in Fig. 2. As is evident from Fig. 2, k_1 and Nu_∞ increase with increase in τ at constant Pe, and hence the heat transfer between the film and the surrounding gas flow is intensified. The minimum value of τ is taken to be -0.75, since when $\tau < -0.75$ return flow is formed close to the free surface, i.e., the condition of unidirectionality is violated. The case when $\tau \to \infty$ corresponds to horizontal flow of the liquid film arising on account of entrainment in the gas flow.

NOTATION

Pe, Peclet number; v, flow velocity, m/sec; T, fluid temperature, K: x, y, coordinates along and transverse to flow, m; a, thermal diffusivity, m^2/sec; Δ, Laplacian, m^{-2}; y_i, coordinates of flow boundary, m; T_0, temperature at input to heat-transfer zone, K; Θ, dimensionless temperature; T^*, temperature used in forming dimensionless parameters, K; Q, fluid flow rate, m^3/sec; S, cross-sectional area of flow, m^2; h, characteristic length transverse to flow, m; δ_{nm}, Kronecker delta; Nu, Nusselt number; Q_1, fluid flow rate per unit width of film, m^2/sec. Indices: i = 1, 2, number of flow boundary.

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